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On Some “Collateral” Effects in the Alpha-convex Theory

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Abstract—Some effects in the α-convex theory of the univalent functions are discussed in the light of the uniqueness problem for the critical point of the conformal radius.

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0. The class of α-convex functions $\mathfrak{M}_\alpha$ is defined for $\alpha \in \mathbb{R}$ as the preimage

$$\mathfrak{M}_\alpha = J_\alpha^{-1}(C) \tag{1}$$

of the Caratheodory class C by means of the operator $J_\alpha = J_\alpha(f,\zeta) = p(\zeta) + \alpha\zeta p'(\zeta)/p(\zeta)$, $p(\zeta) = \zeta f'(\zeta)/f(\zeta)$, acting on the family $\mathcal{LS}$ of all holomorphic functions $f(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots$ in the unit disk $\mathbb{D}$ with $f(\zeta)f'(\zeta)/\zeta \neq 0$, $\zeta \in \mathbb{D}$. The seminal result of the α-theory [1], namely, the inclusion

$$\mathfrak{M}_\alpha \subset S^* \ \text{ (the class of all starlike } f \text{ in } \mathbb{D} \text{ with } f(0) = f'(0) - 1 = 0), \tag{2}$$

have generated the series of works generalizing (2) both in parametrical (see [2–4]) and in functional (e.g., [5, 6]) directions.

1. If we write (2) in the working form

$$J_\alpha(f,\zeta) = \frac{1+\varphi}{1-\varphi}(\zeta), \quad \zeta \in \mathbb{D}, \quad \Rightarrow \quad f \in S^*, \tag{2'}$$

then the illusion can arise that the Schwarz lemma function φ gains the status of the “controlling parameter” for f to be in the class $\mathfrak{M}_\alpha$. But really this is not the case: if $\alpha \in (-1, 0)$, then

$$J_\alpha^{-1}\left(\frac{1+\zeta^2}{1-\zeta^2}\right) \notin \mathfrak{M}_\alpha.$$

Indeed, if $\alpha \neq -1/2$, then

$$J_\alpha^{-1}\left(\frac{1+\zeta^2}{1-\zeta^2}\right) = \zeta + \frac{1}{2\alpha+1}\zeta^3 + \ldots \equiv f_\alpha,$$

whence $|a_3| = |2\alpha+1|^{-1} > 1$, i.e. $|\{f_\alpha, 0\}| > 6$, where $\{f,\zeta\} = (f''/f')'(\zeta) - (f''/f')^2(\zeta)/2$, and the well-known Kraus–Nehari theorem implies $f_\alpha \notin \mathfrak{M}_\alpha$ (by the use of (2)). In the case $\alpha = -1/2$, moreover, the formal expression of f_α contains $\ln\zeta$!

The “solution” of this “phenomenon” is find in more pedantry form of (1), $\mathfrak{M}_\alpha = J^{-1}(C) \cap \mathcal{LS}$: we must keep in mind the domain of definition $\mathcal{LS}$ of the operator J_α. Nevertheless, this “detective” poses the following

Open problem. *Decribe $J_\alpha(\mathfrak{M}_\alpha)$ in C for any $\alpha \in \mathbb{R}$. Find the set of α's such that $J_\alpha(\mathfrak{M}_\alpha) = C$.*

The expression of J_α in terms of φ (see (2′)) presents a some way for the “morphogenesis” of such a description. If we have $\varphi(\zeta) = c\zeta^2 + \ldots$ in (2′), then $a_2 = 0$ and $(2\alpha+1)a_3 = c$. When $\alpha = -1/2$, this

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